Applications of DEs:

Spring problems, II

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Recall from part I, the spring equation

$$mx'' + bx' + kx = F(t)$$

where $x(t)$ denotes the displacement at time t.

Unless otherwise stated, we assume there is no external force: $F(t) = 0$.

The roots of the characteristic polynomial $mD^2 + bD = k = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases:

(a) real distinct roots: in this case the discriminant $b^2 - 4mk$ is positive, so $b^2 > 4mk$. In other words, b is "large". This case is referred to as **overdamped**. In this case, the roots are negative,

$$r_1 = \frac{-b - \sqrt{b^2 - 4mk}}{2m} < 0, \quad \text{and} \quad r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} < 0,$$

so the solution $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is exponentially decreasing.

(b) real repeated roots: in this case the discriminant $b^2 - 4mk$ is zero, so $b = \sqrt{4mk}$. This case is referred to as **critically damped**. This case is said to model new suspension systems in cars [D].

(c) Complex roots: in this case the discriminant $b^2 - 4mk$ is negative, so $b^2 < 4mk$. In other words, b is "small". This case is referred to as **underdamped** (or **simple harmonic** when $b = 0$).

[1] Originally written 9-18-2007.

Example: An 8 lb weight stretches a spring 2 ft. Assume a damping force numerically equal to 2 times the instantaneous velocity acts. Find the displacement at time t, provided that it is released from the equilibrium position with an upward velocity of 3 ft/s. Find the equation of motion and classify the behaviour.

We know $m = 8/32 = 1/4$, $b = 2$, $k = mg/s = 8/2 = 4$, $x(0) = 0$, and $x'(0) = -3$. This means we must solve

$$\frac{1}{4}x'' + 2x' + 4x = 0, \quad x(0) = 0, \quad x'(0) = -3.$$

The roots of the characteristic polynomial are -4 and -4 (so we are in the repeated real roots case), so the general solution is $x(t) = c_1e^{-4t} + c_2te^{-4t}$. The initial conditions imply $c_1 = 0$, $c_2 = -3$, so

$$x(t) = -3te^{-4t}.$$

Using SAGE, we can compute this as well:

SAGE

```
sage: t = var(''t'')
sage: x = function(''x'')
sage: de = lambda y: (1/4)*diff(y,t,t) + 2*diff(y,t) + 4*y
sage: de(x(t))
diff(x(t), t, 2)/4 + 2*diff(x(t), t, 1) + 4*x(t)
sage: desolve(de(x(t)),[x,t])
'(%k2*t+%k1)*%e^-(4*t)'
sage: desolve_laplace(de(x(t)),[''t'',''x''],[0,0,-3])
'-3*t*%e^-(4*t)'
sage: f = lambda t : -3*t*e^(-4*t)
sage: P = plot(f,0,2)
sage: show(P)
```

The graph is shown below.

Example: An 2 kg weight is attached to a spring having spring constant 10. Assume a damping force numerically equal to 4 times the instantaneous velocity acts. Find the displacement at time t, provided that it is released

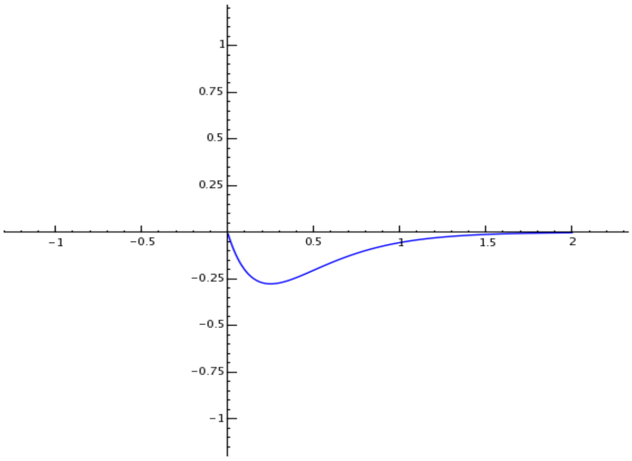

Figure 1: Plot of $(1/4)x'' + 2x' + 4x = 0$, $x(0) = 0$, $x'(0) = -3$, for $0 < t < 2$.

from 1 m below equilibrium with an upward velocity of 1 ft/s. Find the equation of motion and classify the behaviour.

Using SAGE, we can compute this as well:

SAGE

```
sage: t = var(''t'')
sage: x = function(''x'')
sage: de = lambda y: 2*diff(y,t,t) + 4*diff(y,t) + 10*y
sage: desolve_laplace(de(x(t)),["t","x"],[0,1,1])
'%e^-t*(sin(2*t)+cos(2*t))'
sage: desolve_laplace(de(x(t)),["t","x"],[0,1,-1])
'%e^-t*cos(2*t)'
sage: sol = lambda t: e^(-t)*cos(2*t)
sage: P = plot(sol(t),0,2)
sage: show(P)
sage: P = plot(sol(t),0,4)
sage: show(P)
```

The graph is shown below.

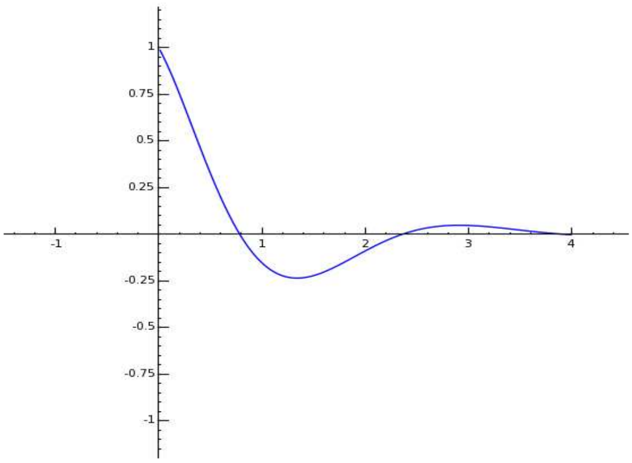

Figure 2: Plot of $2x'' + 4x' + 10x = 0$, $x(0) = 1$, $x'(0) = -1$, for $0 < t < 4$.

Exercise: *Use the problem above. Use* SAGE *to find what time the weight passes through the equilibrium position heading down for the 2nd time.*

Exercise: *An* 2 *kg weight is attached to a spring having spring constant* 10*. Assume a damping force numerically equal to* 4 *times the instantaneous velocity acts. Use* SAGE *to find the displacement at time t, provided that it is released from* 1 *m below equilibrium (with no initial velocity).*

References

[D] Wikipedia entry for damped motion: `http://en.wikipedia.org/wiki/Damping`

[H] General wikipedia introduction to Hooke's Law `http://en.wikipedia.org/wiki/Hookes_law`

[N] Wikipedia entry for Newton's laws of motion (including Newton's 2nd law): `http://en.wikipedia.org/wiki/Newton%27s_laws_of_motion`