Numerical solutions to DEs

Euler's method and improved Euler's method

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Read Euler: he is our master in everything.
- *Pierre Simon de Laplace*

Leonhard Euler was a Swiss mathematician who made significant contributions to a wide range of mathematics and physics including calculus and celestial mechanics (see [Eu1] and [Eu2] for further details).

The goal is to find an approximate solution to the problem

$$y' = f(x, y), \quad y(a) = c, \tag{1}$$

where $f(x, y)$ is some given function. We shall try to approximate the value of the solution at $x = b$, where $b > a$ is given. Sometimes such a method is called "numerically integrating (1)".

Note: the first order DE must be in the form (1) or the method described below does not work. A version of Euler's method for systems of 1-st order DEs and higher order DEs will also be described below.

Euler's method

Geometric idea: The basic idea can be easily expressed in geometric terms. We know the solution, whatever it is, must go through the point (a, c) and we know, at that point, its slope is $m = f(a, c)$. Using the point-slope form of a line, we conclude that the tangent line to the solution curve at (a, c) is (in (x, y)-coordinates, not to be confused with the dependent variable y and independent variable x of the DE)

$$y = c + (x - a)f(a, c).$$

In particular, if $h > 0$ is a given small number (called the **increment**) then taking $x = a + h$ the tangent-line approximation from calculus I gives us:

$$y(a+h) \cong c + h \cdot f(a,c).$$

Now we know the solution passes through a point which is "nearly" equal to $(a+h, c+h \cdot f(a,c)$. We now repeat this tangent-line approximation with (a,c) replaced by $(a+h, c+h \cdot f(a,c)$. Keep repeating this number-crunching at $x = a$, $x = a + h$, $x = a + 2h$, ..., until you get to $x = b$.

Algebraic idea: The basic idea can also be explained "algebraically". Recall from the definition of the derivative in calculus 1 that

$$y'(x) \cong \frac{y(x+h) - y(x)}{h},$$

$h > 0$ is a given and small. This an the DE together give $f(x, y(x)) \cong \frac{y(x+h)-y(x)}{h}$. Now solve for $y(x+h)$:

$$y(x+h) \cong y(x) + h \cdot f(x, y(x)).$$

If we call $h \cdot f(x, y(x))$ the "correction term" (for lack of anything better), call $y(x)$ the "old value of y", and call $y(x+h)$ the "new value of y", then this approximation can be re-expressed

$$y_{new} = y_{old} + h \cdot f(x, y_{old}).$$

Tabular idea: Let $n > 0$ be an integer, which we call the **step size**. This is related to the increment by

$$h = \frac{b-a}{n}.$$

This can be expressed simplest using a table.

x	y	$hf(x,y)$
a	c	$hf(a,c)$
$a+h$	$c+hf(a,c)$	$\vdots$
$a+2h$	$\vdots$	
$\vdots$		
b	???	xxx

The goal is to fill out all the blanks of the table but the xxx entry and find the ??? entry, which is the **Euler's method approximation for** $y(b)$.

Example 1 *Use Euler's method with* $h = 1/2$ *to approximate* $y(1)$*, where*

$$y' - y = 5x - 5, \quad y(0) = 1.$$

Putting the DE into the form (1), we see that here $f(x,y) = 5x + y - 5$*,* $a = 0$*,* $c = 1$*.*

x	y	$hf(x,y) = \frac{5x+y-5}{2}$
0	1	-2
$1/2$	$1 + (-2) = -1$	$-7/4$
1	$-1 + (-7/4) = -11/4$	

so $y(1) \cong -\frac{11}{4} = -2.75$*. This is the final answer.*

Aside: For your information, $y = e^x - 5x$ *solves the DE and* $y(1) = e - 5 = -2.28...$*.*

Here is one way to do this using SAGE *:*

SAGE

```
sage: x,y=PolynomialRing(QQ,2,"xy").gens()
sage: eulers_method(5*x+y-5,1,1,1/3,2)
         x                    y                   h*f(x,y)
         1                    1                   1/3
       4/3                  4/3                     1
       5/3                  7/3                  17/9
         2                 38/9                 83/27
sage: eulers_method(5*x+y-5,0,1,1/2,1,method="none")
[[0, 1], [1/2, -1], [1, -11/4], [3/2, -33/8]]
sage: pts = eulers_method(5*x+y-5,0,1,1/2,1,method="none")
sage: P = plot_
plot_cube          plot_vector_field
sage: P = list_plot(pts)
sage: show(P)
sage: P = line(pts)
sage: show(P)
sage: P1 = list_plot(pts)
sage: P2 = line(pts)
sage: show(P1+P2)
```

The plot is given below.

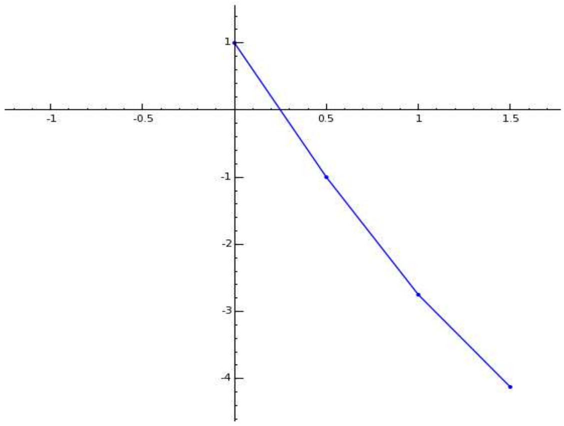

Figure 1: Euler's method with $h = 1/2$ for $x' + x = 1$, $x(0) = 2$.

Improved Euler's method

Geometric idea: The basic idea can be easily expressed in geometric terms. As in Euler's method, we know the solution must go through the point (a, c) and we know its slope there is $m = f(a, c)$. If we went out one step using the tangent line approximation to the solution curve, the approximate slope to the tangent line at $x = a + h, y = c + h \cdot f(a, c)$ would be $m' = f(a + h, c + h \cdot f(a, c))$. The idea is that instead of using $m = f(a, c)$ as the slope of the line to get our first approximation, use $\frac{m+m'}{2}$. The "improved" tangent-line approximation at (a, c) is:

$$y(a + h) \cong c + h \cdot \frac{m + m'}{2} = c + h \cdot \frac{f(a, c) + f(a + h, c + h \cdot f(a, c))}{2}.$$

(This turns out to be a better apprpximation than the tangent-line approximation $y(a + h) \cong c + h \cdot f(a, c)$ used in Euler's method.) Now we know the solution passes through a point which is "nearly" equal to $(a + h, c + h \cdot \frac{m+m'}{2})$. We now repeat this tangent-line approximation with (a, c) replaced by $(a + h, c + h \cdot f(a, c)$. Keep repeating this number-crunching at $x = a$, $x = a + h$, $x = a + 2h$, ..., until you get to $x = b$.

Tabular idea: The integer step size $n > 0$ is related to the increment by

$$h = \frac{b-a}{n},$$

as before.

The improved Euler method can be expressed simplest using a table.

x	y	$h\frac{m+m'}{2} = h\frac{f(x,y)+f(x+h,y+h\cdot f(x,y))}{2}$
a	c	$h\frac{f(a,c)+f(a+h,c+h\cdot f(a,c))}{2}$
$a+h$	$c+h\frac{f(a,c)+f(a+h,c+h\cdot f(a,c))}{2}$	$\vdots$
$a+2h$	$\vdots$	
$\vdots$		
b	???	xxx

The goal is to fill out all the blanks of the table but the xxx entry and find the ??? entry, which is the **improved Euler's method approximation for** $y(b)$.

Example 2 *Use the improved Euler's method with $h = 1/2$ to approximate $y(1)$, where*

$$y' - y = 5x - 5, \quad y(0) = 1.$$

Putting the DE into the form (1), we see that here $f(x,y) = 5x + y - 5$, $a = 0$, $c = 1$. We first compute the "correction term":

$$\begin{aligned} h\frac{f(x,y)+f(x+h,y+h\cdot f(x,y))}{2} &= \frac{5x+y-5+5(x+h)+(y+h\cdot f(x,y))-5}{4} \\ &= \frac{5x+y-5+5(x+h)+(y+h\cdot(5x+y-5))-5}{4} \\ &= 25x/8 + 5y/8 - 5/2. \end{aligned}$$

x	y	$h\frac{m+m'}{2} = \frac{25x+5y-20}{8}$
0	1	$-15/8$
1/2	$1 + (-15/8) = -7/8$	$-95/64$
1	$-7/8 + (-95/64) = -151/64$	

so $y(1) \cong -\frac{151}{64} = -2.35...$ This is the final answer.

Aside: For your information, this is closer to the exact value $y(1) = e - 5 = -2.28...$ than the "usual" Euler's method approximation of -2.75 we obtained above.

Euler's method for systems and higher order DEs

We only sketch the idea in some simple cases. Consider the DE

$$y'' + p(x)y' + q(x)y = f(x), \quad y(a) = e_1, \;\; y'(a) = e_2,$$

and the system

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2), \;\; y_1(a) = c_1, \\ y_2' &= f_2(x, y_1, y_2), \;\; y_2(a) = c_2. \end{aligned}$$

We can treat both cases after first rewriting the DE as a system: create new variables $y_1 = y$ and let $y_2 = y'$. It is easy to see that

$$\begin{aligned} y_1' &= y_2, & y_1(a) &= e_1, \\ y_2' &= f(x) - q(x)y_1 - p(x)y_2, & y_2(a) &= e_2. \end{aligned}$$

Tabular idea: Let $n > 0$ be an integer, which we call the **step size**. This is related to the increment by

$$h = \frac{b-a}{n}.$$

This can be expressed simplest using a table.

x	y_1	$hf_1(x, y_1, y_2)$	y_2	$hf_2(x, y_1, y_2)$
a	e_1	$hf_1(a, e_1, e_2)$	e_2	$hf_2(a, e_1, e_2)$
$a+h$	$e_1 + hf_1(a, e_1, e_2)$	$\vdots$	$e_1 + hf_1(a, e_1, e_2)$	$\vdots$
$a+2h$	$\vdots$			
$\vdots$				
b	???	xxx	xxx	xxx

The goal is to fill out all the blanks of the table but the xxx entry and find the ??? entries, which is the **Euler's method approximation for** $y(b)$.

Example 3 *Using 3 steps of Euler's method, estimate* $x(1)$, *where* $x'' - 3x' + 2x = 1$, $x(0) = 0$, $x'(0) = 1$

First, we rewrite $x'' - 3x' + 2x = 1$, $x(0) = 0$, $x'(0) = 1$, *as a system of* 1^{st} *order DEs with ICs. Let* $x_1 = x$, $x_2 = x'$, *so*

$$
\begin{aligned}
x_1' &= x_2, & x_1(0) &= 0,\\
x_2' &= 1 - 2x_1 + 3x_2, & x_2(0) &= 1.
\end{aligned}
$$

This is the DE rewritten as a system in standard form. (In general, the tabular method applies to any system but it must be in standard form.)

Taking $h = (1-0)/3 = 1/3$*, we have*

t	x_1	$x_2/3$	x_2	$(1-2x_1+3x_2)/3$
0	0	1/3	1	4/3
1/3	1/3	7/9	7/3	22/9
2/3	10/9	43/27	43/9	*xxx*
1	73/27	*xxx*	*xxx*	*xxx*

So $x(1) = x_1(1) \sim 73/27 = 2.7....$

Here is one way to do this using SAGE *:*

SAGE

```
sage: RR = RealField(sci_not=0, prec=4, rnd='RNDU')
sage: t, x, y = PolynomialRing(RR,3,"txy").gens()
sage: f = y; g = 1-2*x+3*y
sage: L = eulers_method_2x2(f,g,0,0,1,1/3,1,method="none")
sage: L
[[0, 0, 1], [1/3, 0.35, 2.5], [2/3, 1.3, 5.5],
 [1, 3.3, 12], [4/3, 8.0, 24]]
sage: eulers_method_2x2(f,g, 0, 0, 1, 1/3, 1)
 t        x         h*f(t,x,y)      y       h*g(t,x,y)
 0        0          0.35           1         1.4
 1/3      0.35       0.88           2.5       2.8
 2/3      1.3        2.0            5.5       6.5
 1        3.3        4.5            12        11
sage: P1 = list_plot([[p[0],p[1]] for p in L])
sage: P2 = line([[p[0],p[1]] for p in L])
sage: show(P1+P2)
```

The plot of the approximation to $x(t)$ is given below.

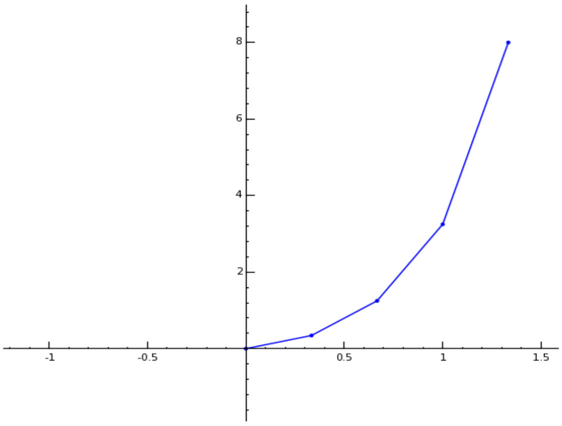

Figure 2: Euler's method with $h = 1/3$ for $x'' - 3x' + 2x = 1$, $x(0) = 0$, $x'(0) = 1$.

Exercise: Use SAGE and Euler's method with $h = 1/3$ for the following problems:

(a) Find the approximate values of $x(1)$ and $y(1)$ where

$$\begin{cases} x' = x + y + t, & x(0) = 0, \\ y' = x - y, & y(0) = 0, \end{cases}$$

(b) Find the approximate value of $x(1)$ where $x' = x^2 + t^2$, $x(0) = 1$.

References

[E] General wikipedia introduction to Euler's method: `http://en.wikipedia.org/wiki/Euler_integration`

[Eu1] Wikipedia entry for Euler: `http://en.wikipedia.org/wiki/Euler`

[Eu2] MacTutor entry for Euler: `http://www-groups.dcs.st-and.ac.uk/%7Ehistory/Biographies/Euler.html`