Solving ODEs:
using Laplace transforms, I

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The Laplace transform (LT) of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\left[f(t)\right](s) = \int_0^\infty e^{-st} f(t)\, dt.$$

This is named for Pierre-Simon Laplace, one of the best French mathematicians in the mid-to-late 18th century [L], [LT]. The LT sends "nice" functions of t (we will be more precise later) to functions of another variable s. It has the wonderful property that it transforms constant-coefficient differential equations in t to algebraic questions in s.

The LT has two very familiar properties: Just as the integral of a sum is the sum of the integrals, the Laplace transform of a sum is the sum of Laplace transforms:

$$\mathcal{L}\left[f(t) + g(t)\right](s) = \mathcal{L}\left[f(t)\right](s) + \mathcal{L}\left[g(t)\right](s)$$

Just as constant factor can be taken outside of an integral, the LT of a constant times a function is that constant times the LT of that function:

$$\mathcal{L}\left[af(t)\right](s) = a\mathcal{L}\left[f(t)\right](s)$$

In other words, the LT is **linear**.

For which functions f is the LT actually defined on? We want the indefinite integral to converge, of course. A function $f(t)$ is of **exponential order** α if there exist constants t_0 and M such that

$$|f(t)| < Me^{\alpha t}, \quad \text{for all } t > t_0.$$

 The graphs were created using SAGE and and GIMP `http://www.gimp.org/` by the author. Originally written 9-25-2007. Last modified 10-1-2007. Some of the latex code is taken from the excellent (public domain!) text by Sean Mauch [M].

Abusing terminology, we say $f(t)$ is of **exponential order** if there is a (finite) constant $\alpha > 0$ for which f is of exponential order α. If $\int_0^{t_0} f(t)\,dt$ exists and $f(t)$ is of exponential order α then the Laplace transform $\mathcal{L}[f](s)$ exists for $s > \alpha$. We shall say that $f(t)$ is "nice", in the sense used above, if it is Riemann-integrable and of exponential order. You can see that the image $F(s)$ of such an $f(t)$ under the Laplace transform is a function of $s > \alpha$ which tends to 0 as $s \to \infty$: for some (possibly huge) constant C, we have $F(s) \le C \int_0^\infty e^{\alpha t} e^{-st}\,dt = \frac{C}{s-\alpha} \to 0$ as $s \to \infty$.

Example 1 *Consider the Laplace transform of $f(t) = 1$. The LT integral converges for $s > 0$.*

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{-st}\,dt \\ &= \left[-\frac{1}{s}e^{-st}\right]_0^\infty \\ &= \frac{1}{s}\end{aligned}$$

Example 2 *Consider the Laplace transform of $f(t) = e^{at}$. The LT integral converges for $s > a$.*

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \left[-\frac{1}{s-a}e^{(a-s)t}\right]_0^\infty \\ &= \frac{1}{s-a}\end{aligned}$$

Define the unit step (Heaviside) function by

$$u(t-c) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t > c, \end{cases}$$

where $c > 0$. This models a power source which turns "on" (1) and "off" (0).

Example 3 *Compute the Laplace transform of the unit step function:*

$$\begin{aligned}\mathcal{L}[u(t-c)](s) &= \int_0^\infty e^{-st}u(t-c)\,dt \\ &= \int_c^\infty e^{-st}\,dt \\ &= \left[\frac{e^{-st}}{-s}\right]_c^\infty \\ &= \frac{e^{-cs}}{s},\end{aligned}$$

for $s > 0$.

The *inverse Laplace transform* is denoted

$$f(t) = \mathcal{L}^{-1}[F(s)](t),$$

where $F(s) = \mathcal{L}\left[f(t)\right](s)$.

Example 4 *Consider*

$$f(t) = \begin{cases} 1, & \text{for } t < 2, \\ 0, & \text{on } t \geq 2. \end{cases}$$

The plot of this function is displayed below:

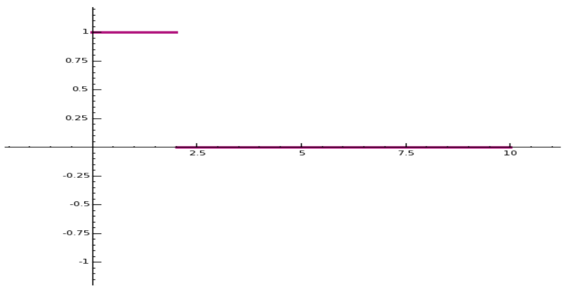

Figure 1: The piecewise constant function $1 - u(t-2)$.

We show how SAGE *can be used to compute the LT of this. (Note this function is "on" then turns "off", whereas the unit step function is "off" then turns "on".*

SAGE

```
sage: t = var('t')
sage: s = var('s')
sage: f = Piecewise([[(0,2),1],[(2,infinity),0]])
sage: f.laplace(t, s)
1/s - e^(-(2*s))/s
sage: f1 = lambda t: 1
sage: f2 = lambda t: 0
sage: f = Piecewise([[(0,2),f1],[(2,10),f2]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5),thickness=3)
sage: show(P)
```

According to SAGE *,*

$$\mathcal{L}[f](s) = 1/s - e^{-2s}/s.$$

Note the function f was redefined for plotting purposes only (the fact that it was redefined over $0 < t < 10$ means that SAGE *will plot it over that range.)*

Next, some properties of the LT.

- Differentiate the definition of the LT with respect to s:

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt.$$

  Repeating this:

$$\frac{d^n}{ds^n} F(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt. \tag{1}$$

  This is called a **derivative theorem** for the LT.

- In the definition of the LT, replace $f(t)$ by it's derivative $f'(t)$:

$$\mathcal{L}[f'(t)](s) = \int_0^\infty e^{-st} f'(t)\, dt.$$

  Now integrate by parts ($u = e^{-st}$, $dv = f'(t)\, dt$):

$$\int_0^\infty e^{-st} f'(t)\, dt = f(t)e^{-st}|_0^\infty - \int_0^\infty f(t)\cdot(-s)\cdot e^{-st}\, dt = -f(0) + s\mathcal{L}[f(t)](s).$$

Therefore, if $F(s)$ is the LT of $f(t)$ then $sF(s) - f(0)$ is the LT of $f'(t)$:

$$\mathcal{L}\left[f'(t)\right](s) = s\mathcal{L}\left[f(t)\right](s) - f(0). \tag{2}$$

- Replace f by f' in (2),

  $$\mathcal{L}\left[f''(t)\right](s) = s\mathcal{L}\left[f'(t)\right](s) - f'(0), \tag{3}$$

  and apply (2) again:

  $$\mathcal{L}\left[f''(t)\right](s) = s^2\mathcal{L}\left[f(t)\right](s) - sf(0) - f'(0), \tag{4}$$

  This, and the previous item, is also called a **derivative theorem** for the LT.

- Using (2) and (4), the LT of any constant coefficient ODE

  $$ax''(t) + bx'(t) + cx(t) = f(t)$$

  is

  $$a(s^2\mathcal{L}\left[x(t)\right](s) - sx(0) - x'(0)) + b(s\mathcal{L}\left[x(t)\right](s) - x(0)) + c\mathcal{L}\left[x(t)\right](s) = F(s),$$

  where $F(s) = \mathcal{L}\left[f(t)\right](s)$. In particular, the LT of the solution, $X(s) = \mathcal{L}\left[x(t)\right](s)$, satisfies

  $$X(s) = \frac{F(s) + asx(0) + ax'(0) + bx(0)}{as^2 + bs + c}.$$

  Note that the denominator is the characteristic polynomial of the DE.

  Moral of the story: it is *always very easy to compute the LT of the solution to any constant coefficient non-homogeneous linear ODE.*

Example 5 *We know now how to compute not only the LT of $f(t) = e^{at}$ (it's $F(s) = (s-a)^{-1}$) but also the LT of any function of the form $t^n e^{at}$ by differentiating it:*

$$\mathcal{L}\left[te^{at}\right] = -F'(s) = (s-a)^{-2}, \quad \mathcal{L}\left[t^2e^{at}\right] = F''(s) = 2\cdot(s-a)^{-3},$$
$$\mathcal{L}\left[t^3e^{at}\right] = -F'''(s) = 2\cdot 3\cdot(s-a)^{-4}, \quad \dots\,,$$

and in general

$$\mathcal{L}\left[t^n e^{at}\right] = (-1)^n F^{(n)}(s) = n! \cdot (s-a)^{-n-1}. \qquad (5)$$

Example 6 *Let us solve the DE*

$$x' + x = t^{100}e^{-t}, \quad x(0) = 0.$$

using LTs. Note this would be highly impractical to solve using undetermined coefficients. (You would have 101 *undetermined coefficients to solve for!)*

First, we compute the LT of the solution to the DE. The LT of the LHS: by (2),

$$\mathcal{L}\left[x' + x\right] = sX(s) + X(s),$$

where $F(s) = \mathcal{L}\left[f(t)\right](s)$*. For the LT of the RHS, let*

$$F(s) = \mathcal{L}\left[e^{-t}\right] = \frac{1}{s+1}.$$

By (1),

$$\frac{d^{100}}{ds^{100}}F(s) = \mathcal{L}\left[t^{100}e^{-t}\right] = \frac{d^{100}}{ds^{100}}\frac{1}{s+1}.$$

The first several derivatives of $\frac{1}{s+1}$ *are as follows:*

$$\frac{d}{ds}\frac{1}{s+1} = -\frac{1}{(s+1)^2}, \quad \frac{d^2}{ds^2}\frac{1}{s+1} = 2\frac{1}{(s+1)^3}, \quad \frac{d^3}{ds^3}\frac{1}{s+1} = -62\frac{1}{(s+1)^4},$$

and so on. Therefore, the LT of the RHS is:

$$\frac{d^{100}}{ds^{100}}\frac{1}{s+1} = 100!\frac{1}{(s+1)^{101}}.$$

Consequently,

$$X(s) = 100!\frac{1}{(s+1)^{102}}.$$

Using (5), we can compute the ILT of this:

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \mathcal{L}^{-1}\left[100!\frac{1}{(s+1)^{102}}\right] = \frac{1}{101}\mathcal{L}^{-1}\left[101!\frac{1}{(s+1)^{102}}\right] = \frac{1}{101}t^{101}e^{-t}.$$

Example 7 *Let us solve the DE*

$$x'' + 2x' + 2x = e^{-2t}, \quad x(0) = x'(0) = 0,$$

using LTs.

The LT of the LHS: by (2) and (3),

$$\mathcal{L}\left[x'' + 2x' + 2x\right] = (s^2 + 2s + 2)X(s),$$

as in the previous example. The LT of the RHS is:

$$\mathcal{L}\left[e^{-2t}\right] = \frac{1}{s+2}.$$

Solving for the LT of the solution algebraically:

$$X(s) = \frac{1}{(s+2)((s+1)^2+1)}.$$

The inverse LT of this can be obtained from LT tables after rewriting this using partial fractions:

$$X(s) = \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{2} \frac{s}{(s+1)^2+1} = \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{2} \frac{s+1}{(s+1)^2+1} + \frac{1}{2} \frac{1}{(s+1)^2+1}.$$

The inverse LT is:

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \frac{1}{2} \cdot e^{-2t} - \frac{1}{2} \cdot e^{-t}\cos(t) + \frac{1}{2} \cdot e^{-t}\sin(t).$$

We show how SAGE *can be used to do some of this.*

SAGE

```
sage: t = var('t')
sage: s = var('s')
sage: f = 1/((s+2)*((s+1)^2+1))
sage: f.partial_fraction()
 1/(2*(s + 2)) - s/(2*(s^2 + 2*s + 2))
sage: f.inverse_laplace(s,t)
 e^(-t)*(sin(t)/2 - cos(t)/2) + e^(-(2*t))/2
```

Exercise: Use SAGE to solve the DE

$$x'' + 2x' + 5x = e^{-t}, \quad x(0) = x'(0) = 0.$$

References

[L] Wikipedia entry for Laplace:
http://en.wikipedia.org/wiki/Pierre-Simon_Laplace

[LT] Wikipedia entry for Laplace transform:
http://en.wikipedia.org/wiki/Laplace_transform

[M] Sean Mauch, *Introduction to methods of Applied Mathematics*,
http://www.its.caltech.edu/~sean/book/unabridged.html