

Solving ODEs: using Laplace transforms, II

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In this lecture, we shall focus on two other aspects of Laplace transforms:

- solving differential equations involving unit step (Heaviside) functions and delta functions,
- convolutions and applications.

It follows from the definition of the LT that if

$$f(t) \xrightarrow{\mathcal{L}} F(s) = \mathcal{L}[f(t)](s),$$

then

$$f(t)u(t-c) \xrightarrow{\mathcal{L}} e^{-cs} \mathcal{L}[f(t+c)](s), \quad (1)$$

and

$$f(t-c)u(t-c) \xrightarrow{\mathcal{L}} e^{-cs} F(s). \quad (2)$$

These two properties are called *translation theorems*.

Example 1 • *First, consider the Laplace transform of the piecewise-defined function $f(t) = (t-1)^2 u(t-1)$. Using (2), this is*

$$\mathcal{L}[f(t)] = e^{-s} \mathcal{L}[t^2](s) = 2 \frac{1}{s^3} e^{-s}.$$

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- Second, consider the Laplace transform of the piecewise-constant function

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ -1 & \text{for } 0 \leq t \leq 2, \\ 1 & \text{for } t > 2. \end{cases}$$

This can be expressed as $f(t) = -u(t) + 2u(t - 2)$, so

$$\begin{aligned} \mathcal{L}[f(t)] &= -\mathcal{L}[u(t)] + 2\mathcal{L}[u(t - 2)] \\ &= -\frac{1}{s} + 2\frac{1}{s}e^{-2s}. \end{aligned}$$

- Finally, consider the Laplace transform of $f(t) = \sin(t)u(t - \pi)$. Using (1), this is

$$\mathcal{L}[f(t)] = e^{-\pi s} \mathcal{L}[\sin(t + \pi)](s) = e^{-\pi s} \mathcal{L}[-\sin(t)](s) = -e^{-\pi s} \frac{1}{s^2 + 1}.$$

The plot of this function $f(t) = \sin(t)u(t - \pi)$ is displayed below:

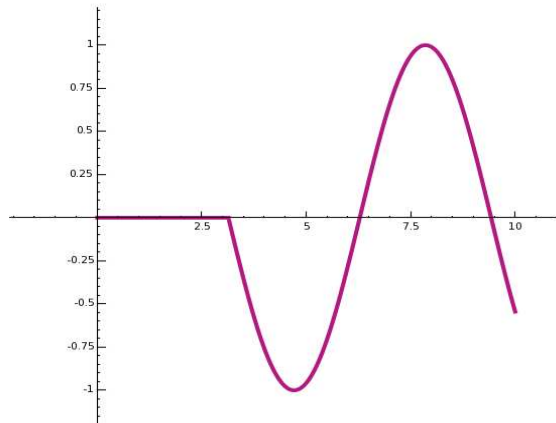


Figure 1: The piecewise continuous function $u(t - \pi) \sin(t)$.

We show how SAGE can be used to compute these LTs.

```

sage: t = var('t')
sage: s = var('s')
sage: assume(s>0)
sage: f = Piecewise([[ (0,1),0],[ (1,infinity),(t-1)^2]])
sage: f.laplace(t, s)
2*e^(-s)/s^3
sage: f = Piecewise([[ (0,2),-1],[ (2,infinity),2]])
sage: f.laplace(t, s)
3*e^(-2*s)/s - 1/s
sage: f = Piecewise([[ (0,pi),0],[ (pi,infinity),sin(t)]])
sage: f.laplace(t, s)
-e^(-(pi*s))/(s^2 + 1)
sage: f1 = lambda t: 0
sage: f2 = lambda t: sin(t)
sage: f = Piecewise([[ (0,pi),f1],[ (pi,10),f2]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5),thickness=3)
sage: show(P)

```

The plot given by these last few commands is displayed above.

Before turning to differential equations, let us introduce convolutions.

Let $f(t)$ and $g(t)$ be continuous (for $t \geq 0$ - for $t < 0$, we assume $f(t) = g(t) = 0$). The *convolution* of $f(t)$ and $g(t)$ is defined by

$$(f * g) = \int_0^t f(u)g(t-u) du = \int_0^t f(t-u)g(u) du.$$

The **convolution theorem** states

$$\mathcal{L}[f * g(t)](s) = F(s)G(s) = \mathcal{L}[f](s)\mathcal{L}[g](s).$$

The LT of the convolution is the product of the LTs. (Or, equivalently, the inverse LT of the product is the convolution of the inverse LTs.)

To show this, do a change-of-variables in the following double integral:

$$\begin{aligned}
\mathcal{L}[f * g(t)](s) &= \int_0^\infty e^{-st} \int_0^t f(u)g(t-u) du dt \\
&= \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u) dt du \\
&= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u) dt du \\
&= \int_0^\infty e^{-su} f(u) du \int_0^\infty e^{-sv} g(v) dv \\
&= \mathcal{L}[f](s)\mathcal{L}[g](s).
\end{aligned}$$

Example 2 Consider the inverse Laplace transform of $\frac{1}{s^3-s^2}$. This can be computed using partial fractions and LT tables. However, it can also be computed using convolutions.

First we factor the denominator, as follows

$$\frac{1}{s^3-s^2} = \frac{1}{s^2} \frac{1}{s-1}.$$

We know the inverse Laplace transforms of each term:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t, \quad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t$$

We apply the convolution theorem:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2} \frac{1}{s-1}\right] &= \int_0^t u e^{t-u} du \\
&= e^t [-ue^{-u}]_0^t - e^t \int_0^t -e^{-u} du \\
&= -t - 1 + e^t
\end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2} \frac{1}{s-1}\right](t) = e^t - t - 1.$$

Example 3 Here is a neat application of the convolution theorem. Consider the convolution

$$f(t) = 1 * 1 * 1 * 1 * 1.$$

What is it? First, take the LT. Since the LT of the convolution is the product of the LTs:

$$\mathcal{L}[1 * 1 * 1 * 1 * 1](s) = (1/s)^5 = \frac{1}{s^5} = F(s).$$

We know from LT tables that $\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = t^4$, so

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \frac{1}{4!}\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = \frac{1}{4!}t^4.$$

Now let us turn to solving a DE of the form

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1. \quad (3)$$

First, take LTs of both sides:

$$as^2Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) = F(s),$$

so

$$Y(s) = \frac{1}{as^2 + bs + c}F(s) + \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}. \quad (4)$$

The function $\frac{1}{as^2 + bs + c}$ is sometimes called the *transfer function* (this is an engineering term) and its inverse LT,

$$w(t) = \mathcal{L}^{-1}\left[\frac{1}{as^2 + bs + c}\right](t),$$

the *weight function* for the DE.

Lemma 4 If $a \neq 0$ then $w(t) = 0$.

(The only proof I have of this is a case-by-case proof using LT tables. Case 1 is when the roots of $as^2 + bs + c = 0$ are real and distinct, case 2 is when the roots are real and repeated, and case 3 is when the roots are

complex. In each case, $w(0) = 0$. The verification of this is left to the reader, if he or she is interested.)

By the above lemma and the first derivative theorem,

$$w'(t) = \mathcal{L}^{-1} \left[\frac{s}{as^2 + bs + c} \right] (t).$$

Using this and the convolution theorem, the inverse LT of (4) is

$$y(t) = (w * f)(t) + ay_0 \cdot w'(t) + (ay_1 + by_0) \cdot w(t). \quad (5)$$

This proves the following fact.

Theorem 5 *The unique solution to the DE (3) is (5).*

Example 6 *Consider the DE $y'' + y = 1$, $y(0) = y'(0) = 1$.*

The weight function is the inverse Laplace transform of $\frac{1}{s^2+1}$, so $w(t) = \sin(t)$. By (5),

$$y(t) = 1 * \sin(t) = \int_0^t \sin(u) du = -\cos(u)|_0^t = 1 - \cos(t).$$

(Yes, it is just that easy!)

If the “impulse” $f(t)$ is piecewise-defined, sometimes the convolution term in the formula (5) is awkward to compute.

Example 7 *Consider the DE $y'' - y' = u(t - 1)$, $y(0) = y'(0) = 0$.*

Taking Laplace transforms gives $s^2Y(s) - sY(s) = \frac{1}{s}e^{-s}$, so

$$Y(s) = \frac{1}{s^3 - s^2} e^{-s}.$$

We know from a previous example that

$$\mathcal{L}^{-1} \left[\frac{1}{s^3 - s^2} \right] (t) = e^t - t - 1,$$

so by the translation theorem (2), we have

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{s^3 - s^2} e^{-s} \right] (t) = (e^{t-1} - (t-1) - 1) \cdot u(t-1) = (e^{t-1} - t) \cdot u(t-1).$$

At present, SAGE [S] lacks the functionality to solve this DE.

Dirac delta functions: The “Dirac delta function” $\delta(t)$ is technically not a function. Roughly speaking, it may be thought of as being analogous to a radar “ping”: if a continuous function $f(t)$ represents an objects’ trajectory then the delta function “samples” its value at $t = 0$. The “graph” of $\delta(t)$ can be visualized as follows:

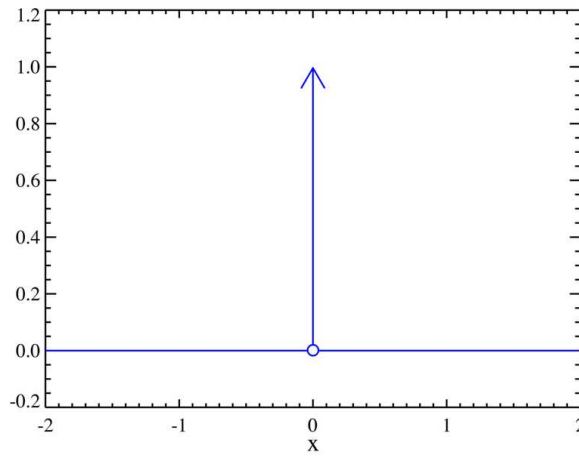


Figure 2: The delta “function”.

To be precise, $\delta(t)$ is a linear functional (that is, a linear function which sends functions to numbers) defined by the formula

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0),$$

where $f(t)$ is any continuous function. Of course, the interval $(-\infty, \infty)$ can be replaced by any interval containing 0. In particular, if $a > 0$ then

$$\int_0^{\infty} \delta(t - a) f(t) e^{-st} dt = f(a) e^{-as}.$$

Though it is not obvious at first glance, the delta function is not unrelated to the unit step function. Indeed, in some sense the delta function may be regarded as the derivative of the (discontinuous!) unit step function. Let $f(t)$ be any function on the real line whose derivative is integrable on $(0, \infty)$ and satisfies $\lim_{t \rightarrow -\infty} f(t) = 0$. Formally integrating by parts, we obtain

$$\begin{aligned} - \int_{-\infty}^{\infty} \delta(t) f(t) dt &= -f(0) = \int_0^{\infty} f'(t) dt = \int_{-\infty}^{\infty} u(t) f'(t) dt \\ &= u(t) f(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(t) f(t) dt = - \int_{-\infty}^{\infty} u'(t) f(t) dt. \end{aligned}$$

Comparing the LHS with the RHS, we see that in some sense (which can be made precise using the theory of functional analysis, but which goes beyond the scope of these notes), we have

$$\delta(t) = u'(t).$$

Example 8 Consider the DE $x'' + x = -\delta(t - \pi)$, $x(0) = x'(0) = 0$. This models a unit mass attached to an undamped spring suspended from a board with no initial displacement or initial velocity. At time $t\pi$, the board is hit very hard (say with a hammer blow) in the upward direction. As we will see, this will start the mass to oscillate.

Taking Laplace transforms gives $s^2X(s) + X(s) = e^{-\pi s}$, so

$$X(s) = -\frac{1}{s^2 + 1}e^{-\pi s}.$$

The inverse LT is $x(t) = \sin(t)u(t - \pi)$ (see Example 1 above, which also has a graph of this solution).

Exercise: (a) Use SAGE to take the LT of $u(t - \pi/4) \cos(t)$.
(b) Use SAGE to compute the convolution $\sin(t) * \cos(t)$.

References

- [D] Wikipedia entry for the Dirac delta function:
http://en.wikipedia.org/wiki/Dirac_delta_function
- [L] Wikipedia entry for Laplace:
http://en.wikipedia.org/wiki/Pierre-Simon_Laplace
- [LT] Wikipedia entry for Laplace transform:
http://en.wikipedia.org/wiki/Laplace_transform
- [M] Sean Mauch, *Introduction to methods of Applied Mathematics*,
<http://www.its.caltech.edu/~sean/book/unabridged.html>
- [S] The SAGE Group, SAGE: *Mathematical software*, version 2.8.6.
<http://sage.scipy.org/>
<http://www.sagemath.org/>